Clay Mathematics Proceedings

An introduction to Heegaard Floer homology

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Contents

1. Introduction

The aim of this paper is to give an introduction to Heegaard Floer homology [**24**] for closed oriented 3-manifolds. We will also discuss a related Floer homology invariant for knots in S^3, [**31**], [**34**].

Let Y be an oriented closed 3-manifold. The simplest version of Heegaard Floer homology associates to Y a finitely generated Abelian

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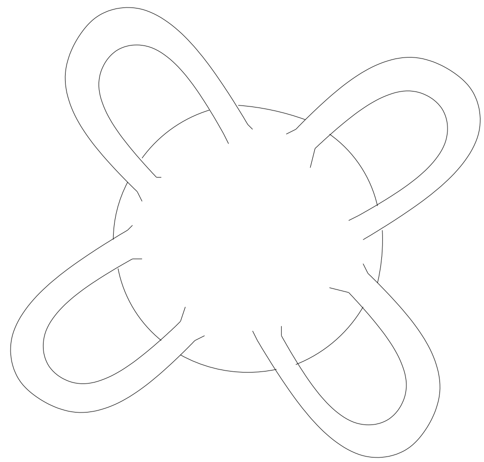

FIGURE 1. A handlebody of genus 4.

group $\widehat{HF}(Y)$. This homology is defined with the help of Heegaard diagrams and Lagrangian Floer homology. Variants of this construction give related invariants $HF^+(Y)$, $HF^-(Y)$, $HF^\infty(Y)$.

While its construction is very different, Heegaard Floer homology is closely related to Seiberg-Witten Floer homology [**10, 15, 17**], and instanton Floer homology [**3, 4, 7**]. In particular it grew out of our attempt to find a more topological description of Seiberg-Witten theory for three-manifolds.

2. Heegaard decompositions and diagrams

Let Y be a closed oriented three-manifold. In this section we describe decompositions of Y into more elementary pieces, called handlebodies.

A *genus g handlebody* U is diffeomorphic to a regular neighborhood of a bouquet of g circles in $\mathbb{R}^3$, see Figure 1. The boundary of U is an oriented surface with genus g. If we glue two such handlebodies together along their common boundary, we get a closed 3-manifold

$$Y = U_0 \cup_\Sigma U_1$$

oriented so that Σ is the oriented boundary of U_0. This is called a Heegaard decomposition for Y.

2.1. Examples. The simplest example is the (genus 0) decomposition of S^3 into two balls. A similar example is given by taking a tubular neighborhood of the unknot in S^3. Since the complement is also a solid torus, we get a genus 1 Heegaard decomposition of S^3.

Other simple examples are given by lens spaces. Take

$$S^3 = \{(z,w) \in \mathbb{C}^2 |\ |z^2| + |w|^2 = 1\}$$

Let $(p,q) = 1$, $1 \leq q < p$. The lens space $L(p.q)$ is given by modding out S^3 with the free $\mathbb{Z}/p$ action

$$f : (z,w) \longrightarrow (\alpha z, \alpha^q w),$$

where $\alpha = e^{2\pi i/p}$. Clearly $\pi_1(L(p,q)) = \mathbb{Z}/p$. Note also that the solid tori $U_0 = |z| \leq \frac{1}{2}$, $U_1 = |z| \geq \frac{1}{2}$ are preserved by the action, and their quotients are also solid tori. This gives a genus 1 Heegaard decomposition of $L(p,q)$.

2.2. Existence of Heegaard decompositions. While the small genus examples might suggest that 3-manifolds that admit Heegaard decompositions are special, in fact the opposite is true:

THEOREM 2.1. *([**39**]) Let Y be an oriented closed three-dimensional manifold. Then Y admits a Heegaard decomposition.*

Proof. Start with a triangulation of Y. The union of the vertecies and the edges gives a graph in Y. Let U_0 be a small neighborhood of this graph. In other words replace each vertex by ball, and each edge by solid cylinder. By definition U_0 is a handlebody. It is easy to see that $Y - U_0$ is also a handlebody, given by a regular neighborhood of a graph on the centers of the triangles and tetrahedrons in the triangulation. □

2.3. Stabilizations. It follows from the above proof that the same three-manifold admits lots of different Heegaard decompositions. In particular, given a Heegaard decomposition $Y = U_0 \cup_\Sigma U_1$ of genus g, we can define another decomposition of genus $g+1$, by choosing two points in Σ and connecting them by a small unknotted arc γ in U_1. Let U_0' be the union of U_0 and a small tubular neghborhood N of γ. Similarly let $U_1' = U_1 - N$. The new decomposition

$$Y = U_0' \cup_{\Sigma'} U_1'$$

is called the *stabilization* of $Y = U_0 \cup_\Sigma U_1$. Clearly $g(\Sigma') = g(\Sigma) + 1$. For an easy example note that the genus 1 decomposition of S^3 described earlier is the stabilization of the genus 0 decomposition.

According to a theorem of Singer [**39**], any two Heegaard decompositions can be connected by stabilizations (and destabilizations):

THEOREM 2.2. *Let (Y, U_0, U_1) and (Y, U_0', U_1') be two Heegaard decompositions of Y with genus g and g' respectively. Then for k large enough the $(k-g')$-fold stabilization of the first decomposition is diffeomorphic with the $(k-g)$-fold stabilization of the second decomposition.*

2.4. Heegaard diagrams. In view of Theorem 2.2 if we find an invariant for Heegaard decompositions with the property that it does not change under stabilization, then this is in fact a three-manifold invariant. For example the Casson invariant [**1, 37**] is defined this way. However for the definition of Heegaard Floer homology we need some additional information which is given by diagrams.

Let us start with a handlebody U of genus g.

DEFINITION 2.3. *A set of attaching circles $(\gamma_1, ..., \gamma_g)$ for U is a collection of closed embedded curves in $\Sigma_g = \partial U$ with the following properties*

- *The curves γ_i are disjoint from each other.*
- *$\Sigma_g - \gamma_1 - \cdots - \gamma_g$ is connected.*
- *The curves γ_i bound disjoint embedded disks in U.*

REMARK 2.4. *The second property in the above definition is equivalent to the property that $([\gamma_1], ..., [\gamma_g])$ are linearly independent in $H_1(\Sigma, \mathbb{Z})$.*

DEFINITION 2.5. *Let (Σ_g, U_0, U_1) be a genus g Heegaard decomposition for Y. A* compatible Heegaard diagram *is given by Σ_g together with a collection of curves $\alpha_1, ..., \alpha_g, \beta_1, ..., \beta_g$ with the property that $(\alpha_1, ..., \alpha_g)$ is a set of attaching circles for U_0 and $(\beta_1, ..., \beta_g)$ is a set of attaching circles for U_1.*

REMARK 2.6. *A Heegaard decomposition of $g > 1$ admits lots of different compatible Heegaard diagrams.*

In the opposite direction any diagram $(\Sigma_g, \alpha_1, ..., \alpha_g, \beta_1, ..., \beta_g)$ where the α and β curves satisfy the first two conditions in Definition 2.3 determine uniquely a Heegaard decomposition and therefore a 3-manifold.

2.5. Examples. It is helpful to look at a few examples. The genus 1 Heegaard decomposition of S^3 corresponds to a diagram $(\Sigma_1, \alpha, \beta)$ where α and β meet transversely in a unique point. $S^1 \times S^2$ corresponds to $(\Sigma_1, \alpha, \alpha)$.

The lens space $L(p, q)$ has a diagram $(\Sigma_1, \alpha, \beta)$ where α and β intersect at p points and in a standard basis $x, y \in H_1(\Sigma_1) = \mathbb{Z} \oplus \mathbb{Z}$, $[\alpha] = y$ and $[\beta] = px + qy$.

Another example is given in Figure 2. Here we think of S^2 as the plane together with the point at infinity. In the picture the two circles

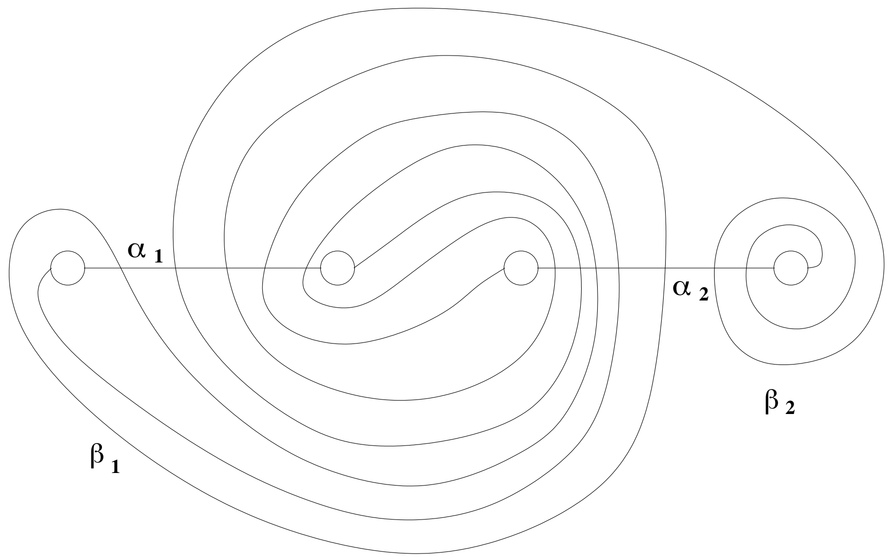

FIGURE 2. A genus 2 Heegaard diagram.

on the left are identified, or equivalently we glue a handle to S^2 along these circles. Similarly we identify the two circles in the right side of the picture. After this identification the two horizontal lines become closed circles α_1 and α_2. As for the two β curves, β_1 lies in the plane and β_2 goes through both handles once.

DEFINITION 2.7. *We can define a one-parameter family of Heegaard diagrams by changing the right side of Figure 2. For $n > 0$ instead of twisting around the right circle two times as in the picture, twist n times. When $n < 0$, twist $-n$ times in the opposite direction. Let Y_n denote the corresponding three-manifold.*

2.6. Heegaard moves. While a Heegaard diagram is a good way to describe Y, the same three-manifold has lots of different diagrams. There are three basic moves on diagrams that do not change the underlying three-manifold. These are *isotopy*, *handle slide* and *stabilization*. The first two moves can be described for attaching circles $\gamma_1, ..., \gamma_g$ for a given handlebody U:

An isotopy moves $\gamma_1, ..., \gamma_g$ in a one parameter family in such a way that the curves remain disjoint.

During handle slide we choose two of the curves, say γ_1 and γ_2, and replace γ_1 with γ_1' provided that γ_1' is any simple, closed curve which is disjoint from the $\gamma_1, \ldots, \gamma_g$ with the property that γ_1', γ_1 and γ_2 bound an embedded pair of pants in $\Sigma - \gamma_3 - \ldots - \gamma_g$ (see Figure 3 for a genus 2 example).

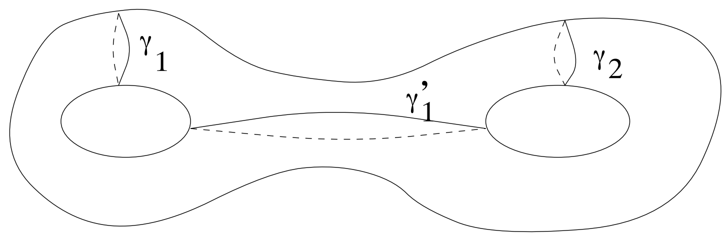

FIGURE 3. Handlesliding γ_1 over γ_2

PROPOSITION 2.8. *([**38**]) Let U be a handlebody of genus g, and let $(\alpha_1, ..., \alpha_g)$, $(\alpha'_1, ..., \alpha'_g)$ be two sets of attaching circles for U. Then the two sets can be connected by a sequence of isotopies and handle slides.*

The stabilization move is defined as follows. We enlarge Σ by making a connected sum with a genus 1 surface $\Sigma' = \Sigma \# E$ and replace $\{\alpha_1, ..., \alpha_g\}$ and $\{\beta_1, ..., \beta_g\}$ by $\{\alpha_1, \ldots, \alpha_{g+1}\}$ and $\{\beta_1, \ldots, \beta_{g+1}\}$ respectively, where α_{g+1} and β_{g+1} are a pair of curves supported in E, meeting transversally in a single point. Note that the new diagram is compatible with the stabilization of the original decomposition.

Combining Theorem 2.2 and Proposition 2.8 we get the following

THEOREM 2.9. *Let Y be a closed oriented 3-manifold. Let*

$$(\Sigma_g, \alpha_1, ..., \alpha_g, \beta_1, ..., \beta_g), \quad (\Sigma_{g'}, \alpha'_1, ..., \alpha'_{g'}, \beta'_1, ..., \beta'_{g'})$$

be two Heegaard diagrams of Y. Then by applying sequences of isotopies, handle slides and stabilizations we can change the above diagrams so that the new diagrams are diffeomorphic to each other.

2.7. The basepoint. In later sections we will also look at pointed Heegaard diagrams $(\Sigma_g, \alpha_1, ..., \alpha_g, \beta_1, ..., \beta_g, z)$, where the basepoint $z \in \Sigma_g$ is chosen in the complement of the curves

$$z \in \Sigma_g - \alpha_1 - ... - \alpha_g - \beta_1 - ... - \beta_g.$$

There is a notion of pointed Heegaard moves. Here we also allow isotopy for the basepoint. During isotopy we require that z is disjoint from the curves. For the pointed handle slide move we require that z is not in the pair of paints region where the handle slide takes place. The following is proved in [**24**].

PROPOSITION 2.10. *Let z_1 and z_2 to be two basepoints. Then the pointed Heegaard diagrams*

$$(\Sigma_g, \alpha_1, ..., \alpha_g, \beta_1, ..., \beta_g, z_1) \text{ and } (\Sigma_g, \alpha_1, ..., \alpha_g, \beta_1, ..., \beta_g, z_2)$$

can be connected by a sequence of pointed isotopies and handle slides.

3. Morse functions and Heegaard diagrams

In this section we study a Morse theoretical approach to Heegaard decompositions. In Morse theory, see [**20**], [**21**], one studies smooth functions on n-dimensional manifolds $f: M^n \to \mathbb{R}$. A point $P \in Y$ is a critical point of f if $\frac{\partial f}{\partial x_i} = 0$ for $i = 1, ..., n$. At a critical point the Hessian matrix $H(P)$ is given by the second partial derivatives $H_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j}$. A critical point P is called non-degenerate if $H(P)$ is non-singular.

DEFINITION 3.1. *The function $f : M^n \to \mathbb{R}$ is called a Morse function if all the critical points are non-degenerate.*

Now suppose that f is a Morse function and P is a critical point. Since $H(P)$ is symmetric, it induces an inner product on the tangent space. The dimension of a maximal negative definite subspace is called the index of P. In other words we can diagonalize $H(P)$ over the reals, and index(P) is the number of negative entries in the diagonal.

Clearly a local minimum of f has index 0, while a local maximum has index n. The local behavior of f around a critical point is studied in [**20**]:

PROPOSITION 3.2. *([**20**]) Let P be an index i critical point of f. Then there is a diffeomorphism h between a neighborhood U of $0 \in \mathbb{R}^n$ and a neighborhood U' of $P \in M^n$ so that*

$$h \circ f = -\sum_{j=1}^{i} x_j^2 + \sum_{j=i+1}^{n} x_j^2.$$

For us it will be benefital to look at a special class of Morse functions:

DEFINITION 3.3. *A Morse function f is called self-indexing if for each critical point P we have $f(P) = \text{index}(P)$.*

PROPOSITION 3.4. [**20**] *Every smooth n-dimensional manifold M admits a self-indexing Morse function. Furthermore if M is connected and has no boundary, then we can choose f so that it has unique index 0 and index n critical points.*

The following exercises can be proved by studying how the level sets $f^{-1}((\infty, t])$ change when t goes through a critical value.

EXERCISE 3.5. *If $f : Y \longrightarrow [0, 3]$ is a self-indexing Morse function on Y with one minimum and one maximum, then f induces a Heegaard decomposition with Heegaard surface $\Sigma = f^{-1}(3/2)$, and handlebodies $U_0 = f^{-1}[0, 3/2]$, $U_1 = f^{-1}[3/2, 3]$.*

Exercise 3.6. *Show that if* Σ *has genus* g*, then* f *has* g *index one and* g *index two critical points.*

Let us denote the index 1 and 2 critical points of f by $P_1, ..., P_g$ and $Q_1, ..., Q_g$ respectively.

Lemma 3.7. *The Morse function and a Riemannian metric on* Y *induces a Heegaard diagram for* Y*.*

Proof. Take the gradient vector field ∇f of the Morse function. For each point $x \in \Sigma$ we can look at the gradient trajectory of $\pm\nabla f$ that goes through x. Let α_i denote the set of points that flow down to the critical point P_i and let β_i correspond to the points that flow up to Q_i. It follows from Proposition 3.2 and the fact that f is self indexing that α_i, β_i are simple closed curves in Σ. It is also easy to see that $\alpha_1, ..., \alpha_g$ and $\beta_1, ..., \beta_g$ are attaching circles for U_0 and U_1 respectively. It follows that this is a Heegaard diagram of Y compatible to the given Heegaard decomposition. □

4. Symmetric products and totally real tori

For a pointed Heegaard diagram $(\Sigma_g, \alpha_1, ..., \alpha_g, \beta_1, ..., \beta_g, z)$ we can associate certain configuration spaces that will be used in later sections in the definition of Heegaard Floer homology. Our ambient space is

$$\mathrm{Sym}^g(\Sigma_g) = \Sigma_g \times \cdots \times \Sigma_g / S_g \ ,$$

where S_g denotes the symmetry group on g letters In other words $\mathrm{Sym}^g(\Sigma_g)$ consists of unordered g-tuple of points in Σ_g where the same points can appear more than one times. Although S_g does not act freely, $\mathrm{Sym}^g(\Sigma_g)$ is a smooth manifold. Furthermore a complex structure on Σ_g induces a complex structure on $\mathrm{Sym}^g(\Sigma_g)$.

The topology of symmetric products of surfaces is studied in [**16**].

Proposition 4.1. *Let* Σ *be a genus* g *surface. Then* $\pi_1(\mathrm{Sym}^g(\Sigma)) \cong H_1(\mathrm{Sym}^g(\Sigma)) \cong H_1(\Sigma)$*.*

Proposition 4.2. *Let* Σ *be a Riemann surface of genus* $g > 2$*, then*

$$\pi_2(\mathrm{Sym}^g(\Sigma)) \cong \mathbb{Z}.$$

The generator of $S \in \pi_2(\mathrm{Sym}^g(\Sigma))$ can be constructed in the following way: Take a hyperelliptic involution τ on Σ, then $(y, \tau(y), z, ..., z)$ is a sphere representing S. An explicit calculation gives

LEMMA 4.3. *Let $S \in \pi_2(\mathrm{Sym}^g(\Sigma))$ be the positive generator as above. Then*

$$\langle c_1(\mathrm{Sym}^g(\Sigma_g)), [S] \rangle = 1$$

REMARK 4.4. *The small genus examples can be understood as well. When $g = 1$ we get a torus and π_2 is trivial.* $\mathrm{Sym}^2(\Sigma_2)$ *is diffeomorphic to the real four-dimensional torus blown up at one point. Here π_2 is large but after dividing with the action of $\pi_1(\mathrm{Sym}^2(\Sigma_2))$ we get*

$$\pi_2'(\mathrm{Sym}^2(\Sigma_2)) \cong \mathbb{Z}$$

with the generator S as before. $\langle c_1, [S] \rangle = 1$ still holds.

EXERCISE 4.5. *Compute $\pi_2(\mathrm{Sym}^2(\Sigma_2))$.*

4.1. Totally real tori, and V_z. Inside $\mathrm{Sym}^g(\Sigma_g)$ our attaching circles induce a pair of smoothly embedded, g-dimensional tori

$$\mathbb{T}_\alpha = \alpha_1 \times ... \times \alpha_g \quad \text{and} \quad \mathbb{T}_\beta = \beta_1 \times ... \times \beta_g \ .$$

More precisely $\mathbb{T}_\alpha$ consists of those g-tuples of points $\{x_1, ..., x_g\}$ for which $x_i \in \alpha_i$ for $i = 1, ..., g$.

These tori enjoy a certain compatibility with any complex structure on $\mathrm{Sym}^g(\Sigma)$ induced from Σ:

DEFINITION 4.6. *Let (Z, J) be a complex manifold, and $L \subset Z$ be a submanifold. Then, L is called* totally real *if none of its tangent spaces contains a J-complex line, i.e. $T_\lambda L \cap J T_\lambda L = (0)$ for each $\lambda \in L$.*

EXERCISE 4.7. *Let $\mathbb{T}_\alpha \subset \mathrm{Sym}^g(\Sigma)$ be the torus induced from a set of attaching circles $\alpha_1, ..., \alpha_g$. Then, $\mathbb{T}_\alpha$ is a totally real submanifold of* $\mathrm{Sym}^g(\Sigma)$ *(for any complex structure induced from Σ).*

The basepoint z also induce a subspace that we use later:

$$V_z = \{z\} \times \mathrm{Sym}^{g-1}(\Sigma_g),$$

which has complex codimension 1. Note that since z is in the complement of the α and β curves, V_z is disjoint from $\mathbb{T}_\alpha$ and $\mathbb{T}_\beta$.

We finish the section with the following problems.

EXERCISE 4.8. *Show that*

$$\frac{H_1(\mathrm{Sym}^g(\Sigma))}{H_1(\mathbb{T}_\alpha) \oplus H_1(\mathbb{T}_\beta)} \cong \frac{H_1(\Sigma)}{[\alpha_1], ..., [\alpha_g], [\beta_1], ..., [\beta_g]} \cong H_1(Y; \mathbb{Z}).$$

EXERCISE 4.9. *Compute $H_1(Y_n, \mathbb{Z})$ for the three-manifolds Y_n in Definition 2.7.*

5. Disks in symmetric products

Let $\mathbb{D}$ be the unit disk in $\mathbb{C}$. Let e_1, e_2 be the arcs in the boundary of $\mathbb{D}$ with $\mathrm{Re}(z) \geq 0$, $\mathrm{Re}(z) \leq 0$ respectively.

Definition 5.1. *Given a pair of intersection points* $\mathbf{x}, \mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, *a Whitney disk connecting* $\mathbf{x}$ *and* $\mathbf{y}$ *is a continuos map*

$$u: \ \mathbb{D} \longrightarrow \mathrm{Sym}^g(\Sigma_g)$$

with the properties that $u(-i) = \mathbf{x}$, $u(i) = \mathbf{y}$, $u(e_1) \subset \mathbb{T}_\alpha$, $u(e_2) \subset \mathbb{T}_\beta$. *Let* $\pi_2(\mathbf{x}, \mathbf{y})$ *denote the set of homotopy classes of maps connecting* $\mathbf{x}$ *and* $\mathbf{y}$.

The set $\pi_2(\mathbf{x}, \mathbf{y})$ is equipped with a certain multiplicative structure. Note that there is a way to splice spheres to disks:

$$\pi_2'(\mathrm{Sym}^g(\Sigma)) * \pi_2(\mathbf{x}, \mathbf{y}) \longrightarrow \pi_2(\mathbf{x}, \mathbf{y}).$$

Also, if we take a disk connecting $\mathbf{x}$ to $\mathbf{y}$, and one connecting $\mathbf{y}$ to $\mathbf{z}$, we can glue them, to get a disk connecting $\mathbf{x}$ to $\mathbf{z}$. This operation gives rise to a multiplication

$$*\colon \pi_2(\mathbf{x}, \mathbf{y}) \times \pi_2(\mathbf{y}, \mathbf{z}) \longrightarrow \pi_2(\mathbf{x}, \mathbf{z}).$$

5.1. An obstruction.

Let $\mathbf{x}, \mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ be a pair of intersection points. Choose a pair of paths $a\colon [0,1] \longrightarrow \mathbb{T}_\alpha$, $b\colon [0,1] \longrightarrow \mathbb{T}_\beta$ from $\mathbf{x}$ to $\mathbf{y}$ in $\mathbb{T}_\alpha$ and $\mathbb{T}_\beta$ respectively. The difference $a - b$, gives a loop in $\mathrm{Sym}^g(\Sigma)$.

Definition 5.2. *Let* $\epsilon(\mathbf{x}, \mathbf{y})$ *denote the image of* $a - b$ *in* $H_1(Y, Z)$ *under the map given by Exercise 4.8. Note that* $\epsilon(\mathbf{x}, \mathbf{y})$ *is independent of the choice of the paths* a *and* b.

It is obvious from the definition that if $\epsilon(\mathbf{x}, \mathbf{y}) \neq 0$ then $\pi_2(\mathbf{x}, \mathbf{y})$ is empty. Note that ϵ can be calculated in Σ, using the identification between $\pi_1(\mathrm{Sym}^g(\Sigma))$ and $H_1(\Sigma)$. Specifically, writing $\mathbf{x} = \{x_1, \ldots, x_g\}$ and $\mathbf{y} = \{y_1, \ldots, y_g\}$, we can think of the path $a\colon [0,1] \longrightarrow \mathbb{T}_\alpha$ as a collection of arcs in $\alpha_1 \cup \ldots \cup \alpha_g \subset \Sigma$, whose boundary is given by $\partial a = y_1 + \ldots + y_g - x_1 - \ldots - x_g$; similarly, the path $b\colon [0,1] \longrightarrow \mathbb{T}_\beta$ can be viewed as a collection of arcs in $\beta_1 \cup \ldots \cup \beta_g \subset \Sigma$, whose boundary is given by $\partial b = y_1 + \ldots + y_g - x_1 - \ldots - x_g$. Thus, the difference $a - b$ is a closed one-cycle in Σ, whose image in $H_1(Y; \mathbb{Z})$ is the difference $\epsilon(\mathbf{x}, \mathbf{y})$ defined above.

Clearly ϵ is additive, in the sense that

$$\epsilon(\mathbf{x}, \mathbf{y}) + \epsilon(\mathbf{y}, \mathbf{z}) = \epsilon(\mathbf{x}, \mathbf{z}).$$

Definition 5.3. *Partition the intersection points of* $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$ *into equivalence classes, where* $\mathbf{x} \sim \mathbf{y}$ *if* $\epsilon(\mathbf{x}, \mathbf{y}) = 0$.

Exercise 5.4. *Take a genus* 1 *Heegaard diagram of* $L(p, q)$*, and isotope* α *and* β *so that they have only* p *intersection points. Show that all the intersection points lie in different equivalence classes.*

Exercise 5.5. *In the genus* 2 *example of Figure 2 find all the intersection points in* $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$*, (there are* 18 *of them), and partition the points into equivalence classes (there are* 2 *equivalence classes).*

5.2. Domains. In order to understand topological disks in $\mathrm{Sym}^g(\Sigma_g)$ it is helpful to study their "shadow" in Σ_g.

Definition 5.6. *Let* $\mathbf{x}, \mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$*. For any point* $w \in \Sigma$ *which is in the complement of the* α *and* β *curves let*

$$n_w : \pi_2(\mathbf{x}, \mathbf{y}) \longrightarrow \mathbb{Z}$$

denote the algebraic intersection number

$$n_w(\phi) = \#\phi^{-1}(\{w\} \times \mathrm{Sym}^{g-1}(\Sigma_g)).$$

Note that since $V_w = \{w\} \times \mathrm{Sym}^{g-1}(\Sigma_g)$ is disjoint from $\mathbb{T}_\alpha$ and $\mathbb{T}_\beta$, n_w is well-defined.

Definition 5.7. *Let* $D_1, \dots, D_m$ *denote the closures of the components of* $\Sigma - \alpha_1 - \dots - \alpha_g - \beta_1 - \dots - \beta_g$*. Given* $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$ *the* domain associated to ϕ *is the formal linear combination of the regions* $\{D_i\}_{i=1}^m$*:*

$$\mathcal{D}(\phi) = \sum_{i=1}^{m} n_{z_i}(\phi) D_i,$$

where $z_i \in D_i$ *are points in the interior of* D_i*. If all the coefficients* $n_{z_i}(\phi) \geq 0$*, then we write* $\mathcal{D}(\phi) \geq 0$.

Exercise 5.8. *Let* $\mathbf{x}, \mathbf{y}, \mathbf{p} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$*,* $\phi_1 \in \pi_2(\mathbf{x}, \mathbf{y})$ *and* $\phi_2 \in \pi_2(\mathbf{y}, \mathbf{p})$*. Show that*

$$\mathcal{D}(\phi_1 * \phi_2) = \mathcal{D}(\phi_1) + \mathcal{D}(\phi_2).$$

Similarly

$$\mathcal{D}(S * \phi) = \mathcal{D}(\phi) + \sum_{i=1}^{n} D_i \ ,$$

where S *denotes the positive generator of* $\pi_2(\mathrm{Sym}^g(\Sigma_g))$.

The domain $\mathcal{D}(\phi)$ can be regarded as a two-chain. In the next exercise we study its boundary.

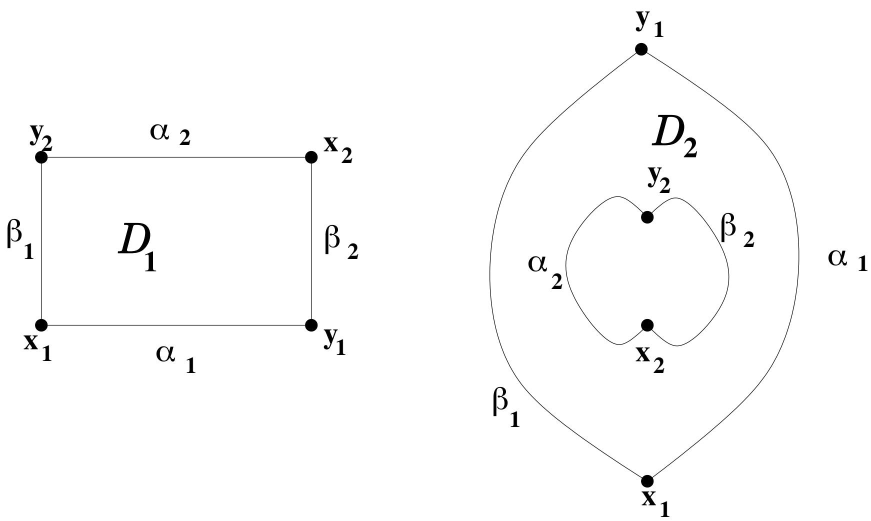

FIGURE 4. Domains of disks in $\mathrm{Sym}^2(\Sigma)$

EXERCISE 5.9. *Let* $\mathbf{x} = (x_1, ..., x_g)$, $\mathbf{y} = (y_1, ..., y_g)$ *where*

$$x_i \in \alpha_i \cap \beta_i, \quad y_i \in \alpha_i \cap \beta_{\sigma^{-1}(i)}$$

and σ *is a permutation. For* $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$, *show that*

- *The restriction of* $\partial\mathcal{D}(\phi)$ *to* α_i *is a one-chain with boundary* $y_i - x_i$.
- *The restriction of* $\partial\mathcal{D}(\phi)$ *to* β_i *is a one-chain with boundary* $x_i - y_{\sigma(i)}$.

REMARK 5.10. *Informally the above result says that* $\partial(\mathcal{D}(\phi))$ *connects* x *to* y *on* α *curves and* y *to* x *on* β *curves.*

EXERCISE 5.11. *Take the genus 2 examples is of Figure 4. Find disks* ϕ_1 *and* ϕ_2 *with* $\mathcal{D}(\phi_1) = D_1$ *and* $\mathcal{D}(\phi_2) = D_2$.

DEFINITION 5.12. *Let* $\mathbf{x}, \mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$. *If a formal sum*

$$\mathcal{A} = \sum_{i=1}^{n} a_i D_i$$

satisfies that $\partial\mathcal{A}$ *connects* $\mathbf{x}$ *to* $\mathbf{y}$ *along* α *curves and connects* $\mathbf{y}$ *to* $\mathbf{x}$ *along the* β *curves, we will say that* $\partial\mathcal{A}$ *connects* $\mathbf{x}$ *to* $\mathbf{y}$.

When $g > 1$ the argument in Exercise 5.9 can be reversed:

PROPOSITION 5.13. *Suppose that* $g > 1$, $\mathbf{x}, \mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$. *If* $\mathcal{A}$ *connects* $\mathbf{x}$ *to* $\mathbf{y}$ *then there is a homotopy class* $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$ *with*

$$\mathcal{D}(\phi) = \mathcal{A}$$

Furthermore if $g > 2$ then ϕ is uniquely determined by $\mathcal{A}$.

As an easy corollary we have the following

PROPOSITION 5.14. [**24**] *Suppose $g > 2$. For each $\mathbf{x}, \mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, if $\epsilon(\mathbf{x}, \mathbf{y}) \neq 0$, then $\pi_2(\mathbf{x}, \mathbf{y})$ is empty; otherwise,*

$$\pi_2(\mathbf{x}, \mathbf{y}) \cong \mathbb{Z} \oplus H^1(Y, \mathbb{Z}).$$

REMARK 5.15. *When $g = 2$ we can define $\pi_2'(\mathbf{x}, \mathbf{y})$ by modding out $\pi_2(\mathbf{x}, \mathbf{y})$ with the relation: ϕ_1 is equivalent to ϕ_2 if $\mathcal{D}(\phi_1) = \mathcal{D}(\phi_2)$. For $\epsilon(\mathbf{x}, \mathbf{y}) = 0$ we have*

$$\pi_2'(\mathbf{x}, \mathbf{y}) \cong \mathbb{Z} \oplus H^1(Y, \mathbb{Z}).$$

Note that working with π_2' is the same as working with homology classes *of disks, and for simplifying notation this is the approach used in* [**25**].

6. Spin^c-structures

In order to refine the discussion about the equivalence classes encountered in the previous section we will need the notion of Spin^c structures. These structures can be defined in every dimension. For three-dimensional manifolds it is convenient to use a reformulation of Turaev [**40**].

Let Y be an oriented closed 3-manifold. Since Y has trivial Euler characteristic, it admits nowhere vanishing vector fields.

DEFINITION 6.1. *Let v_1 and v_2 be two nowhere vanishing vector fields. We say that v_1 is homologous to v_2 if there is a ball B in Y with the property that $v_1|_{Y-B}$ is homotopic to $v_2|_{Y-B}$. This gives an equivalence relation, and we define the space of* Spin^c *structures over Y as nowhere vanishing vector fields modulo this relation.*

We will denote the space of Spin^c structures over Y by Spin$^c(Y)$.

6.1. Action of $H^2(Y, \mathbb{Z})$ on Spin$^c(Y)$. Fix a trivialization τ of the tangent bundle TY. This gives a one-to-one correspondence between vector fields v over Y and maps f_v from Y to S^2.

DEFINITION 6.2. *Let μ denote the positive generator of $H^2(S^2, \mathbb{Z})$. Define*

$$\delta^\tau(v) = f_v^*(\mu) \in H^2(Y, \mathbb{Z})$$

EXERCISE 6.3. *Show that δ^τ induces a one-to-one correspondence between* Spin$^c(Y)$ *and $H^2(Y, \mathbb{Z})$.*

The map δ^τ is independent of the the trivialization if $H_1(Y, \mathbb{Z})$ has no two-torsion. In the general case we have a weaker property:

EXERCISE 6.4. *Show that if* v_1 *and* v_2 *are a pair of nowhere vanishing vector fields over* Y*, then the difference*

$$\delta(v_1, v_2) = \delta^\tau(v_1) - \delta^\tau(v_2) \in H^2(Y, \mathbb{Z})$$

is independent of the trivialization τ*, and*

$$\delta(v_1, v_2) + \delta(v_2, v_3) = \delta(v_1, v_3).$$

This gives an action of $H^2(Y, \mathbb{Z})$ on $\mathrm{Spin}^c(Y)$. If $a \in H^2(Y, \mathbb{Z})$ and $v \in \mathrm{Spin}^c(Y)$ we define $a + v \in \mathrm{Spin}^c(Y)$ by the property that $\delta(a + v, v) = a$. Similarly for $v_1, v_2 \in \mathrm{Spin}^c(Y)$, we let $v_1 - v_2$ denote $\delta(v_1, v_2)$.

There is a natural involution on the space of Spin^c structures which carries the homology class of the vector field v to the homology class of $-v$. We denote this involution by the map $\mathfrak{s} \mapsto \overline{\mathfrak{s}}$.

There is also a natural map

$$c_1 \colon \mathrm{Spin}^c(Y) \longrightarrow H^2(Y, \mathbb{Z}),$$

the first Chern class. This is defined by $c_1(\mathfrak{s}) = \mathfrak{s} - \overline{\mathfrak{s}}$. It is clear that $c_1(\overline{\mathfrak{s}}) = -c_1(\mathfrak{s})$.

6.2. Intersection points and Spin^c **structures.** Now we are ready to define a map

$$s_z : \ \mathbb{T}_\alpha \cap \mathbb{T}_\beta \longrightarrow \mathrm{Spin}^c(Y),$$

which will be a refinement of the equivalence classes given by $\epsilon(\mathbf{x}, \mathbf{y})$:

Let f be a Morse function on Y compatible with the attaching circles $\alpha_1, ..., \alpha_g$, $\beta_1, ..., \beta_g$. Then each $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ determines a g-tuple of trajectories for ∇f connecting the index one critical points to index two critical points. Similarly z gives a trajectory connecting the index zero critical point with the index three critical point. Deleting tubular neighborhoods of these $g+1$ trajectories, we obtain the complement of disjoint union of balls in Y where the gradient vector field ∇f does not vanish. Since each trajectory connects critical points of different parities, the gradient vector field has index 0 on all the boundary spheres, so it can be extended as a nowhere vanishing vector field over Y. According to our definition of Spin^c-structures the homology class of the nowhere vanishing vector field obtained in this manner gives a Spin^c structure. Let us denote this element by $s_z(\mathbf{x})$. The following is proved [**24**].

LEMMA 6.5. *Let* $\mathbf{x}, \mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$*. Then we have*

$$s_z(\mathbf{y}) - s_z(\mathbf{x}) = \mathrm{PD}[\epsilon(\mathbf{x}, \mathbf{y})]. \tag{1}$$

In particular $s_z(\mathbf{x}) = s_z(\mathbf{y})$ *if and only if* $\pi_2(\mathbf{x}, \mathbf{y})$ *is non-empty.*

EXERCISE 6.6. *Let $(\Sigma_1, \alpha, \beta)$ be a genus 1 Heegaard diagram of $L(p, 1)$ so that α and β have p intersection points. Using this diagram $\Sigma_1 - \alpha - \beta$ has p components. Choose a point z_i in each region. Show that for any $x \in \alpha \cap \beta$, we have*

$$s_{z_i}(x) \neq s_{z_j}(x)$$

for $i \neq j$.

7. Holomorphic disks

A complex structure on Σ induces a complex structure on $\mathrm{Sym}^g(\Sigma_g)$. For a given homotopy class $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$ let $\mathcal{M}(\phi)$ denote the moduli space of holomorphic representatives of ϕ. Note that in order to guarantee that $\mathcal{M}(\phi)$ is smooth, in Lagrangian Floer homology one has to use appropriate perturbations, see [**8**], [**9**], [**11**].

The moduli space $\mathcal{M}(\phi)$ admit an $\mathbb{R}$ action. This corresponds to complex automorphisms of the unit disk that preserve i and $-i$. It is easy to see that this group is isomorphic to $\mathbb{R}$. For example using the Riemann mapping theorem change the unit disk to the infinite strip $[0, 1] \times i\mathbb{R} \subset \mathbb{C}$, where e_1 corresponds to $1 \times i\mathbb{R}$ and e_2 corresponds $0 \times i\mathbb{R}$. Then the automorphisms preserving e_1 and e_2 correspond the vertical translations. Now if $u \in \mathcal{M}(\phi)$ then we could precompose u with any of these automorphisms and get another holomorphic disk. Since in the definition of the boundary map we would like to count holomorphic disks we will divide $\mathcal{M}(\phi)$ by the above $\mathbb{R}$ action, and define the unparametrized moduli space

$$\widehat{\mathcal{M}}(\phi) = \frac{\mathcal{M}(\phi)}{\mathbb{R}}.$$

It is easy to see that the $\mathbb{R}$ action is free except in the case when ϕ is the homotopy class of the constant map ($\phi \in \pi_2(\mathbf{x}, \mathbf{x})$, with $\mathcal{D}(\phi) = 0$). In this case $\mathcal{M}(\phi)$ is a single point corresponding to the constant map.

The moduli space $\mathcal{M}(\phi)$ has an expected dimension called the Maslov index $\mu(\phi)$, see [**35**], which corresponds to the index of an elliptic operator. The Maslov index has the following significance: If we vary the almost complex structure of $\mathrm{Sym}^g(\Sigma_g)$ in an n-dimensional family, the corresponding parametrized moduli space has dimension $n + \mu(\phi)$ around solutions that are smoothly cut out by the defining equation. The Maslov index is additive:

$$\mu(\phi_1 * \phi_2) = \mu(\phi_1) + \mu(\phi_2)$$

and for the homotopy class of the constant map μ is equal to zero.

Lemma 7.1. *([**24**]) Let $S \in \pi'_2(\mathrm{Sym}^g(\Sigma))$ be the positive generator. Then for any $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$, we have that*

$$\mu(\phi + k[S]) = \mu(\phi) + 2k.$$

Proof. It follows from the excision principle for the index that attaching a topological sphere Z to a disk changes the Maslov index by $2\langle c_1, [Z]\rangle$ (see [**18**]). On the other hand for the positive generator S we have $\langle c_1, [S]\rangle = 1$. □

Corollary 7.2. *If $g = 2$ and $\phi, \phi' \in \pi_2(\mathbf{x}, \mathbf{y})$ satisfies*

$$\mathcal{D}(\phi) = \mathcal{D}(\phi')$$

then $\mu(\phi) = \mu(\phi')$. In particular μ is well-defined on $\pi'_2(\mathbf{x}, \mathbf{y})$.

Lemma 7.3. *If $\mathcal{M}(\phi)$ is non-empty, then $\mathcal{D}(\phi) \geq 0$.*

Proof. Let us choose a reference point z_i in each region $\mathcal{D}_i$. Since V_{z_i} is a subvariety, a holomorphic disk is either contained in it (which is excluded by the boundary conditions) or it must meet it non-negatively. □

By studying energy bounds, orientations and Gromov limits we prove in [**24**]

Theorem 7.4. *There is a family of (admissible) perturbations with the property that if $\mu(\phi) = 1$ then $\widehat{\mathcal{M}}(\phi)$ is a compact oriented zero dimensional manifold. When $g = 2$, the same result holds for $\phi \in \pi'_2(\mathbf{x}, \mathbf{y})$ as well.*

7.1. Examples. The space of holomorphic disks connecting $\mathbf{x}, \mathbf{y}$ can be given an alternate description, using only maps between one-dimensional complex manifolds.

Lemma 7.5. *([**24**]) Given any holomorphic disk $u \in \mathcal{M}(\phi)$, there is a g-fold branched covering space $p\colon \widehat{\mathbb{D}} \longrightarrow \mathbb{D}$ and a holomorphic map $\widehat{u}\colon \widehat{\mathbb{D}} \longrightarrow \Sigma$, with the property that for each $z \in \mathbb{D}$, $u(z)$ is the image under $\widehat{u}$ of the pre-image $p^{-1}(z)$.*

Exercise 7.6. *Let ϕ_1, ϕ_2 be homotopy classes in Figure 4, with $\mathcal{D}(\phi_1) = D_1$, $\mathcal{D}(\phi_2) = D_2$. Also let $\phi_0 \in \pi_2(\mathbf{y}, \mathbf{x})$ be a class with $\mathcal{D}(\phi_0) = -D_1$. Show that $\mu(\phi_1) = 1$, $\mu(\phi_2) = 0$ and $\mu(\phi_0) = -1$.*

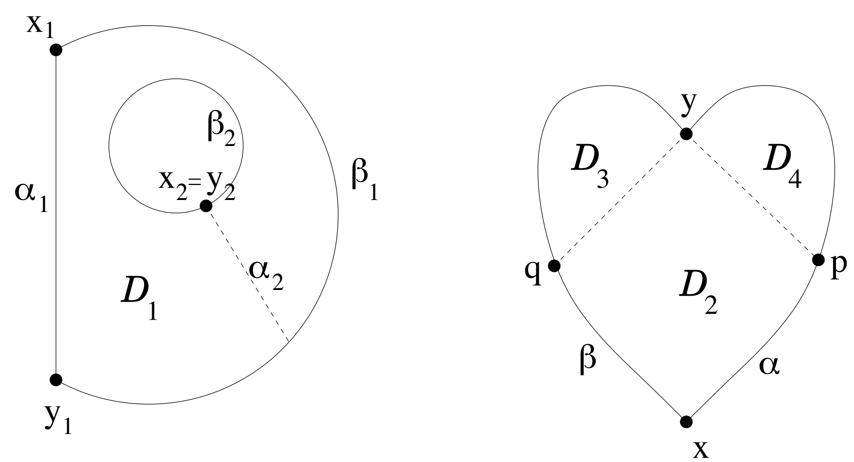

FIGURE 5.

For additional examples see Figure 5. The left example is in the second symmetric product and $x_2 = y_2$. The right example is in the first symmetric product, the α and β curve intersect each other in 4 points. Let ϕ_3, ϕ_4 be classes with $\mathcal{D}(\phi_3) = D_1$, $\mathcal{D}(\phi_4) = D_2 + D_3 + D_4$.

EXERCISE 7.7. *Show that* $\mu(\phi_3) = 1$ *and* $\mu(\phi_4) = 2$.

EXERCISE 7.8. *Use the Riemman mapping theorem to show that* $\widehat{\mathcal{M}}(\phi_4)$ *is homeomorphic to an open intervall* $\mathcal{I}$.

EXERCISE 7.9. *Study the limit of* $u_i \in \mathcal{I}$ *as* u_i *approaches one of the ends in* $\mathcal{I}$. *Show that the limit corresponds to a decomposition*

$$\phi_4 = \phi_5 * \phi_6, \text{ or } \phi_4 = \phi_7 * \phi_8,$$

where $\mathcal{D}(\phi_5) = D_2 + D_4$, $\mathcal{D}(\phi_6) = D_3$, $\mathcal{D}(\phi_7) = D_2 + D_3$ *and* $\mathcal{D}(\phi_8) = D_4$.

8. The Floer chain complexes

In this section we will define the various chain complexes corresponding to $\widehat{HF}$, HF^+, HF^- and HF^∞.

We start with the case when Y is a rational homology 3-sphere. Let $(\Sigma, \alpha_1, ..., \alpha_g, \beta_1, ..., \beta_g, z)$ be a pointed Heegaard diagram with genus $g > 0$ for Y. Choose a Spin^c structure $t \in \text{Spin}^c(Y)$.

Let $\widehat{CF}(\boldsymbol{\alpha}, \boldsymbol{\beta}, t)$ denote the free Abelian group generated by the points in $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ with $s_z(\mathbf{x}) = t$. This group can be endowed with a relative grading

$$\text{gr}(\mathbf{x}, \mathbf{y}) = \mu(\phi) - 2n_z(\phi), \tag{2}$$

where ϕ is any element $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$, and μ is the Maslov index.

In view of Proposition 5.14 and Lemma 7.1, this integer is independent of the choice of homotopy class $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$.

DEFINITION 8.1. *Choose a perturbation as in Theorem 7.4. For $\mathbf{x}, \mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ and $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$ let us define $c(\phi)$ to be the signed number of points in $\widehat{\mathcal{M}}(\phi)$, if $\mu(\phi) = 1$. If $\mu(\phi) \neq 1$ let $c(\phi) = 0$.*

Let

$$\partial\colon \widehat{CF}(\boldsymbol{\alpha}, \boldsymbol{\beta}, t) \longrightarrow \widehat{CF}(\boldsymbol{\alpha}, \boldsymbol{\beta}, t)$$

be the map defined by:

$$\partial \mathbf{x} = \sum_{\{\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta,\ \phi \in \pi_2(\mathbf{x}, \mathbf{y}) \big| s_z(\mathbf{y}) = t,\ n_z(\phi) = 0\}} c(\phi) \cdot \mathbf{y}$$

By analyzing the Gromov compactification of $\widehat{\mathcal{M}}(\phi)$ for $n_z(\phi) = 0$ and $\mu(\phi) = 2$ it is proved in [**24**] that $(\widehat{CF}(\boldsymbol{\alpha}, \boldsymbol{\beta}, t), \partial)$ is a chain complex; i.e. $\partial^2 = 0$.

DEFINITION 8.2. *The Floer homology groups $\widehat{HF}(\boldsymbol{\alpha}, \boldsymbol{\beta}, t)$ are the homology groups of the complex $(\widehat{CF}(\boldsymbol{\alpha}, \boldsymbol{\beta}, t), \partial)$.*

One of the main results of [**24**] is that the homology group $\widehat{HF}(\boldsymbol{\alpha}, \boldsymbol{\beta}, t)$ is independent of the Heegaard diagram, the basepoint and the other choices in the definition (complex structures, perturbations). After analyzing the effect of isotopies, handle slides and stabilizations, it is proved in [**24**] that under pointed isotopies, pointed handle slides, and stabilizations we get chain homotopy equivalent complexes $\widehat{CF}(\boldsymbol{\alpha}, \boldsymbol{\beta}, t)$. This together with Theorem 2.9, and Proposition 2.10 imples:

THEOREM 1. *([**24**]) Let $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$ and $(\Sigma', \boldsymbol{\alpha}', \boldsymbol{\beta}', z')$ be pointed Heegaard diagrams of Y, and $t \in \mathrm{Spin}^c(Y)$. Then the homology groups $\widehat{HF}(\boldsymbol{\alpha}, \boldsymbol{\beta}, t)$ and $\widehat{HF}(\boldsymbol{\alpha}', \boldsymbol{\beta}', t)$ are isomorphic.*

Using the above theorem we can at last define $\widehat{HF}$:

$$\widehat{HF}(Y, t) = \widehat{HF}(\boldsymbol{\alpha}, \boldsymbol{\beta}, t).$$

8.1. $CF^\infty(Y)$**.** The definition in the previous section uses the basepoint z in a special way: in the boundary map we only count holomorphic disks that are disjoint from the subvariety V_z.

Now we study a chain complex where all the holomorphic disks are used (but we still record the intersection number with V_z):

Let $CF^\infty(\boldsymbol{\alpha},\boldsymbol{\beta},t)$ be the free Abelian group generated by pairs $[\mathbf{x},i]$ where $s_z(\mathbf{x})=t$, and $i\in\mathbb{Z}$ is an integer. We give the generators a relative grading defined by

$$\mathrm{gr}([\mathbf{x},i],[\mathbf{y},j])=\mathrm{gr}(\mathbf{x},\mathbf{y})+2i-2j.$$

Let

$$\partial\colon CF^\infty(\boldsymbol{\alpha},\boldsymbol{\beta},t)\longrightarrow CF^\infty(\boldsymbol{\alpha},\boldsymbol{\beta},t)$$

be the map defined by:

$$\partial[\mathbf{x},i]=\sum_{\mathbf{y}\in\mathbb{T}_\alpha\cap\mathbb{T}_\beta}\ \sum_{\phi\in\pi_2(\mathbf{x},\mathbf{y})} c(\phi)\cdot[\mathbf{y},i-n_z(\phi)]. \tag{3}$$

There is an isomorphism U on $CF^\infty(\boldsymbol{\alpha},\boldsymbol{\beta},t)$ given by

$$U([\mathbf{x},i])=[\mathbf{x},i-1]$$

that decreases the grading by 2.

It is proved in [**23**] that for rational homology three-spheres $HF^\infty(Y,t)$ is always isomorphic to $\mathbb{Z}[U,U^{-1}]$. So clearly this is not an interesting invariant. Luckily the base-point z together with Lemma 7.3 induces a filtration on $CF^\infty(\boldsymbol{\alpha},\boldsymbol{\beta},t)$ and that produces more subtle invariants.

8.2. $CF^+(\boldsymbol{\alpha},\boldsymbol{\beta},t)$ **and** $CF^-(\boldsymbol{\alpha},\boldsymbol{\beta})$**.** Let $CF^-(\boldsymbol{\alpha},\boldsymbol{\beta},t)$ denote the subgroup of $CF^\infty(\boldsymbol{\alpha},\boldsymbol{\beta},t)$ which is freely generated by pairs $[\mathbf{x},i]$, where $i<0$. Let $CF^+(\boldsymbol{\alpha},\boldsymbol{\beta},t)$ denote the quotient group

$$CF^\infty(\boldsymbol{\alpha},\boldsymbol{\beta},t)/CF^-(\boldsymbol{\alpha},\boldsymbol{\beta},t)$$

LEMMA 8.3. *The group $CF^-(\boldsymbol{\alpha},\boldsymbol{\beta},t)$ is a subcomplex of $CF^\infty(\boldsymbol{\alpha},\boldsymbol{\beta},t)$, so we have a short exact sequence of chain complexes:*

$$0\longrightarrow CF^-(\boldsymbol{\alpha},\boldsymbol{\beta},t)\xrightarrow{\ \iota\ }CF^\infty(\boldsymbol{\alpha},\boldsymbol{\beta},t)\xrightarrow{\ \pi\ }CF^+(\boldsymbol{\alpha},\boldsymbol{\beta},t)\longrightarrow 0.$$

Proof. If $[\mathbf{y},j]$ appears in $\partial([\mathbf{x},i])$ then there is a homotopy class $\phi(\mathbf{x},\mathbf{y})$ with $\mathcal{M}(\phi)$ non-empty, and $n_z(\phi)=i-j$. According to Lemma 7.3 we have $\mathcal{D}(\phi)\geq 0$ and in particular $i\geq j$. □

Clearly, U restricts to an endomorphism of $CF^-(\boldsymbol{\alpha},\boldsymbol{\beta},t)$ (which lowers degree by 2), and hence it also induces an endomorphism on the quotient $CF^+(\boldsymbol{\alpha},\boldsymbol{\beta},t)$.

EXERCISE 8.4. *There is a short exact sequence*

$$0\longrightarrow \widehat{CF}(\boldsymbol{\alpha},\boldsymbol{\beta},t)\xrightarrow{\ \iota\ }CF^+(\boldsymbol{\alpha},\boldsymbol{\beta},t)\xrightarrow{\ U\ }CF^+(\boldsymbol{\alpha},\boldsymbol{\beta},t)\longrightarrow 0,$$

where $\iota(\mathbf{x})=[\mathbf{x},0]$.

Definition 8.5. *The Floer homology groups* $HF^+(\boldsymbol{\alpha},\boldsymbol{\beta},t)$ *and* $HF^-(\boldsymbol{\alpha},\boldsymbol{\beta},t)$ *are the homology groups of* $(CF^+(\boldsymbol{\alpha},\boldsymbol{\beta},t),\partial)$ *and* $(CF^-(\boldsymbol{\alpha},\boldsymbol{\beta},t),\partial)$ *respectively.*

It is proved in [**24**] that the chain homotopy equivalences under pointed isotopies, handle slides and stabilizations for $\widehat{CF}$ can be lifted to filtered chain homotopy equivalences on CF^∞ and in particular the corresponding Floer homologies are unchanged. This allows us to define

$$HF^\pm(Y,t) = HF^\pm(\boldsymbol{\alpha},\boldsymbol{\beta},t).$$

8.3. Three manifolds with $b_1(Y) > 0$. When $b_1(Y)$ is positive, then there is a technical problem due to the fact that $\pi_2(\mathbf{x},\mathbf{y})$ is larger. In definition of the boundary map we have now infinitely many homotopy classes with Maslov index 1. In order to get a finite sum we have to prove that only finitely many of these homotopy classes support holomorphic disks. This is achieved through the use of special Heegaard diagrams together with the positivity property of Lemma 7.3, see [**24**]. With this said, the constructions from the previous subsections apply and give the Heegaard Floer homology groups. The only difference is that when the image of $c_1(t)$ in $H^2(Y,\mathbb{Q})$ is non-zero, the Floer homologies no longer have relative $\mathbb{Z}$ grading.

9. A few examples

We study Heegaard Floer homology for a few examples. To simplify things we deal with homology three spheres. Here $H_1(Y,\mathbb{Z}) = 0$ so there is a unique Spin^c-structure. In [**27**] we show how to use maps on $HF^\pm$ induced by smooth cobordisms to lift the relative grading to absolute grading.

For $Y = S^3$ we can use a genus 1 Heegaard diagram. Here α and β intersect each other in a unique point $\mathbf{x}$. It follows that CF^+ is generated $[\mathbf{x},i]$ with $i \geq 0$. Since $\mathrm{gr}[\mathbf{x},i]-\mathrm{gr}[\mathbf{x},i-1] = 2$, the boundary map is trivial so $HF^+(S^3)$ is isomorphic with $\mathbb{Z}[U,U^{-1}]/\mathbb{Z}[U]$ as a $\mathbb{Z}[U]$ module. The absolute grading is determined by

$$\mathrm{gr}([\mathbf{x},0]) = 0.$$

A large class of homology three-spheres is provided by Brieskorn spheres: Recall that if p, q, and r are pairwise relatively prime integers, then the Brieskorn variety $V(p,q,r)$ is the locus

$$V(p,q,r) = \{(x,y,z) \in \mathbb{C}^3 \big| x^p + y^q + z^r = 0\}$$

DEFINITION 9.1. *The Brieskorn sphere $\Sigma(p,q,r)$ is the homology sphere obtained by $V(p,q,r)\cap S^5$ (where $S^5\subset\mathbb{C}^3$ is the standard 5-sphere).*

The simplest example is the Poincare sphere $\Sigma(2,3,5)$.

EXERCISE 9.2. *Show that the diagram in Definition 2.7 with $n=3$ is a Heegaard diagram for $\Sigma(2,3,5)$.*

Unfortunately in this Heegaard diagram there are lots of generators (21) and computing the Floer chain complex directly is not an easy task. Instead of this direct approach one can study how the Heegaard Floer homologies change when the three-manifold is modified by surgeries along knots. In [**27**] we use this surgery exact sequences to prove

PROPOSITION 9.3.

$$HF^+_k(\Sigma(2,3,5))=\begin{cases}\mathbb{Z} & \textit{if } k \textit{ is even and } k\geq 2\\ 0 & \textit{otherwise}\end{cases}$$

Moreover,

$$U\colon HF^+_{k+2}(\Sigma(2,3,5))\longrightarrow HF^+_k(\Sigma(2,3,5))$$

is an isomorphism for $k\geq 2$.

This means that as a relatively graded $Z[U]$ module $HF^+((\Sigma(2,3,5))$ is isomorphic to $HF^+(S^3)$, but the absolute grading still distinguishes them.

Another example is provided by $\Sigma(2,3,7)$. (Note that this three manifold corresponds to the $n=5$ diagram when we switch the role of the α and β circles.)

PROPOSITION 9.4.

$$HF^+_k(\Sigma(2,3,7))=\begin{cases}\mathbb{Z} & \textit{if } k \textit{ is even and } k\geq 0\\ \mathbb{Z} & \textit{if } k=-1\\ 0 & \textit{otherwise}\end{cases}\tag{4}$$

For a description of $HF^+(\Sigma(p,q,r))$ see [**29**], and also [**22**], [**36**].

10. Knot Floer homology

In this section we study a version of Heegaard Floer homology that can be applied to knots in three-manifolds. Here we will restrict our attention to knots in S^3. For a more general discussion see [**31**] and [**34**].

Let us consider a Heegaard diagram $(\Sigma_g,\alpha_1,...,\alpha_g,\beta_1,...,\beta_g)$ for S^3 equipped with two basepoints w and z. This data gives rise to a knot in S^3 by the following procedure. Connect w and z by a curve a in $\Sigma_g - \alpha_1 - ... - \alpha_g$ and also by another curve b in $\Sigma_g - \beta_1 - ... - \beta_g$. By pushing a and b into U_0 and U_1 respectively, we obtain a knot $K \subset S^3$. We call the data $(\Sigma_g, \boldsymbol{\alpha}, \boldsymbol{\beta}, w, z)$ a two-pointed Heegaard diagram compatible with the knot K.

A Morse theoretic interpretation can be given as follows. Fix a metric on Y and a self-indexing Morse function so that the induced Heegaard diagram is $(\Sigma_g, \alpha_1, ..., \alpha_g, \beta_1, ..., \beta_g)$. Then the basepoints w, z give two trajectories connecting the index 0 and index 3 critical points. Joining these arcs together gives the knot K.

LEMMA 10.1. *Every knot can be represented by a two-pointed Heegaard diagram.*

Proof. Fix a height function h on K so that for the two critical points A and B, we have $h(A) = 0$ and $h(B) = 3$. Now extend h to a self-indexing Morse function from $K \subset Y$ to Y so that the index 1 and 2 critical points are disjoint from K, and let z and w be the two intersection points of K with the Heegaard surface $\tilde{h}^{-1}(3/2)$. □

A straightforward generalization of $\widehat{CF}$ is the following.

DEFINITION 10.2. *Let K be a knot in S^3 and $(\Sigma_g, \alpha_1, ..., \alpha_g, \beta_1, ..., \beta_g, z, w)$ be a compatible two-pointed Heegaard diagram. Let $C(K)$ be the free abelian group generated by the intersection points $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$. For a generic choice of almost complex structures let $\partial_K : C(K) \longrightarrow C(K)$ be given by*

$$\partial_K(\mathbf{x}) = \sum_{\mathbf{y}} \sum_{\{\phi \in \pi_2(\mathbf{x},\mathbf{y}) | \mu(\phi)=1,\ n_z(\phi)=n_w(\phi)=0\}} c(\phi) \cdot \mathbf{y} \tag{5}$$

PROPOSITION 10.3. *([**31**], [**34**]) $(C(K), \partial_K)$ is a chain complex. Its homology $H(K)$ is independent of the choice of two-pointed Heegaard diagrams representing K, and the almost complex structures.*

10.1. Examples. For the unknot U we can use the standard genus 1 Heegaard diagram of S^3, and get $H(U) = \mathbb{Z}$.

EXERCISE 10.4. *Take the two-pointed Heegaard diagram in Figure 6. Show that the corresponding knot is trefoil $T_{2,3}$.*

EXERCISE 10.5. *Find all the holomorphic disks in Figure 6, and show that $H(T_{2,3})$ has rank 3.*

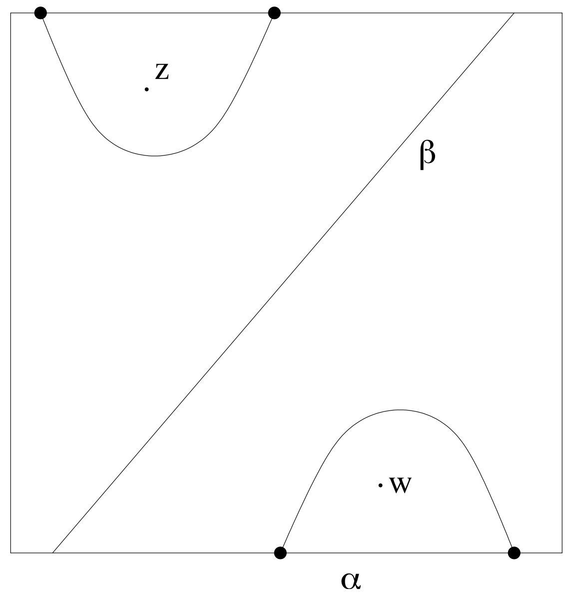

FIGURE 6.

10.2. A bigrading on $C(K)$. For $C(K)$ we define two gradings. These correspond to functions:

$$F,\ G : \mathbb{T}_\alpha \cap \mathbb{T}_\beta \longrightarrow \mathbb{Z}.$$

We start with F:

DEFINITION 10.6. *For $\mathbf{x}, \mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ let*

$$f(\mathbf{x}, \mathbf{y}) = n_z(\phi) - n_w(\phi),$$

where $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$.

EXERCISE 10.7. *Show that for $\mathbf{x}, \mathbf{y}, \mathbf{p} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ we have*

$$f(\mathbf{x}, \mathbf{y}) + f(\mathbf{y}, \mathbf{p}) = f(\mathbf{x}, \mathbf{p}).$$

EXERCISE 10.8. *Show that f can be lifted uniquely to a function $F\colon \mathbb{T}_\alpha \cap \mathbb{T}_\beta \longrightarrow \mathbb{Z}$ satisfying the relation*

$$F(\mathbf{x}) - F(\mathbf{y}) = f(x, y), \tag{6}$$

and the additional symmetry

$$\#\{\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta \big| F(\mathbf{x}) = i\} \equiv \#\{\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta \big| F(\mathbf{x}) = -i\} \pmod 2$$

for all $i \in \mathbb{Z}$

The other grading comes from the Maslov grading.

Definition 10.9. *For* $\mathbf{x}, \mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ *let*

$$g(\mathbf{x}, \mathbf{y}) = \mu(\phi) - 2n_w(\phi),$$

where $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$.

In order to lift g to an absolute grading we use the one-pointed Heegaard diagram $(\Sigma_g, \boldsymbol{\alpha}, \boldsymbol{\beta}, w)$. This is a Heegaard diagram of S^3. It follows that the homology of $\widehat{CF}(\mathbb{T}_\alpha, \mathbb{T}_\beta, w)$ is isomorphic to $\mathbb{Z}$. Using the normalization that this homology is supported in grading zero we get a function

$$G\colon \mathbb{T}_\alpha \cap \mathbb{T}_\beta \longrightarrow \mathbb{Z}$$

that associates to each intersection points its absolute grading in $\widehat{CF}(\mathbb{T}_\alpha, \mathbb{T}_\beta, w)$. It also follows that $G(\mathbf{x}) - G(\mathbf{y}) = g(\mathbf{x}, \mathbf{y})$.

Definition 10.10. *Let* $C_{i,j}$ *denote the free Abelian group generated by those intersection points* $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ *that satisfy*

$$i = F(\mathbf{x}), \quad j = G(\mathbf{x}).$$

The following is straightforward:

Lemma 10.11. *For a two-pointed Heegaard diagram corresponding to a knot* K *in* S^3 *decompose* $C(K)$ *as*

$$C(K) = \bigoplus_{i,j} C_{i,j}.$$

Then $\partial_K(C_{i,j})$ *is contained in* $C_{i,j-1}$.

As a corollary we can decompose $H(K)$:

$$H(K) = \bigoplus_{i,j} H_{i,j}(K).$$

Since the chain homotopy equivalences of $C(K)$ induced by (two-pointed) Heegaard moves respects both gradings it follows that $H_{i,j}(K)$ is also a knot invariant.

11. Kauffman states

When studying knot Floer homology it is natural to consider a special diagram where the intersection points correspond to Kauffman states.

Let K be a knot in S^3. Fix a projection for K. Let $v_1, ..., v_n$ denote the double points in the projection. If we forget the pattern of over and under crossings in the diagram we get an immersed circle C in the plane.

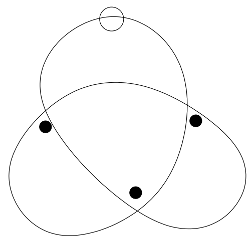

FIGURE 7.

FIGURE 8. **The definition of $a(c_i)$ for both kinds of crossings.**

Fix an edge e which appears in the closure of the unbounded region A in the planar projection. Let B be the region on the other side of the marked edge.

DEFINITION 11.1. *([**14**]) A Kauffman state (for the projection and the distinguished edge e) is a map that associates for each double point v_i one of the four corners in such a way that each component in $S^2 - C - A - B$ gets exactly one corner.*

Let us write a Kauffman state as $(c_1, ..., c_n)$, where c_i is a corner for v_i.

For an example see Figure 7 that shows the Kauffman states for the trefoil. In that picture the black dots denote the corners, and the white circle indicates the marking.

EXERCISE 11.2. *Find the Kauffman states for the $T_{2,2n+1}$ torus knots, (using a projection with $2n+1$ double points).*

11.1. Kauffman states and Alexander polynomial. The Kauffman states could be used to compute the Alexander polynomial for the knot K. Fix an orientation for K. Then for each corner c_i we define $a(c_i)$ by the formula in Figure 8, and $B(c_i)$ by the formula in Figure 9.

THEOREM 11.3. *([**14**]) Let K be knot in S^3, and fix an oriented projection of K with a marked edge. Let $\mathcal{K}$ denote the set of Kauffman*

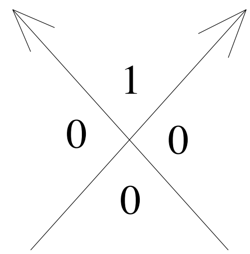

FIGURE 9. **The definition of $B(c_i)$.**

states for the projection. Then the polynomial

$$\sum_{c \in \mathcal{K}} \prod_{i=1}^{n} (-1)^{B(c_i)} T^{a(c_i)}$$

is equal to the symmetrized Alexander polynomial $\Delta_K(T)$ of K.

12. Kauffman states and Heegaard diagrams

PROPOSITION 12.1. *Let K be a knot and S^3. Fix a knot projection for K together with a marked edge. Then there is a Heegaard diagram for K, where the generators are in one-to-one correspondence with Kauffman states of the projection.*

Proof. Let C be the immersed circle as before. A regular neighborhood $nd(C)$ is a handlebody of genus $n+1$. Clearly $S^3 - nd(C)$ is also a handlebody, so we get a Heegaard decomposition of S^3. Let Σ be the oriented boundary of $S^3 - nd(C)$. This will be the Heegaard surface. The complement of C in the plane has $n+2$ components. For each region, except for A, we associate an α curve, which is the intersection of the region with Σ. It is easy to see $\Sigma - \alpha_1 - ... - \alpha_{n+1}$ is connected and all α_i bound disjoint disks in $S^3 - nd(C)$.

Fix a point in the edge e and let β_{n+1} be the meridian for K around this point. The curves $\beta_1, ..., \beta_n$ correspond to the double points $v_1, ..., v_n$, see Figure 10. As for the basepoints, choose w and z on the two sides of β_{n+1}. There is a small arc connecting z and w. This arc is in the complement of the α curves. We can also choose a long arc from w to z in the complement of the β curves that travels along the knot K. It follows that this two-pointed Heegaard diagram is compatible to K.

In order to see the relation between $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$ and Kauffman states note that in a neighborhood of each v_i, there are at most four intersection points of β_i with circles corresponding to the four regions which

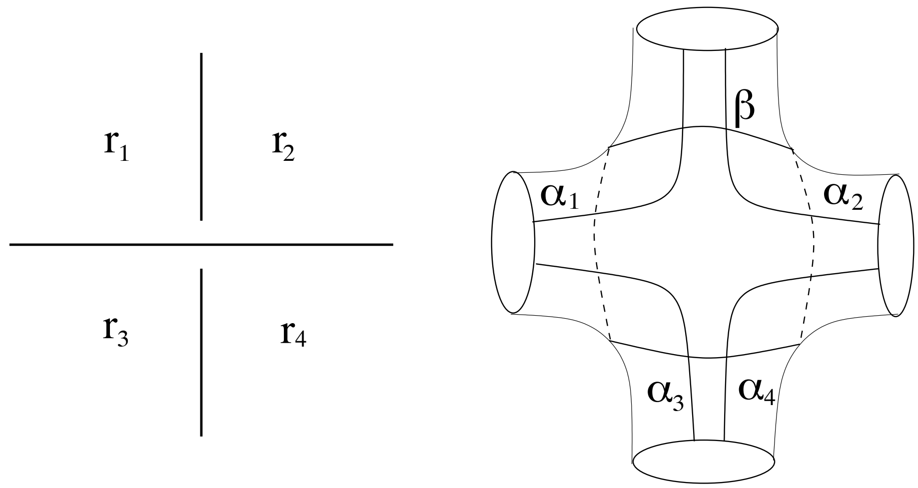

FIGURE 10. **Special Heegaard diagram for knot crossings.** At each crossing as pictured on the left, we construct a piece of the Heegaard surface on the right (which is topologically a four-punctured sphere). The curve β is the one corresponding to the crossing on the left; the four arcs $\alpha_1, ..., \alpha_4$ will close up.

contain v_i, see Figure 10. Clearly these intersection points are in one-to-one correspondence with the corners. This property together with the observation that β_{n+1} intersects only the α curve of region B finishes the proof. □

13. A combinatorial formula

In this section we describe $F(\mathbf{x})$ and $G(\mathbf{x})$ in terms of the knot projection. Both of these function will be given as a state sum over the corners of the corresponding Kauffman state. For a given corner c_i we use $a(c_i)$ and $b(c_i)$, where $a(c_i)$ given as before, see Figure 8, and $b(c_i)$ is defined in Figure 11. Note that $b(c_i)$ and $B(c_i)$ are congruent modulo 2. The following result is proved in [**28**].

THEOREM 13.1. *Fix an oriented knot projection for K together with a distinguished edge. Let us fix a two-pointed Heegaard diagram for K as above. For $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ let $(c_1, ..., c_n)$ be the corresponding Kauffman state. Then we have*

$$F(\mathbf{x}) = \sum_{i=1}^{n} a(c_i) \qquad G(\mathbf{x}) = \sum_{i=1}^{n} b(c_i).$$

FIGURE 11. **Definition of $b(c_i)$.**

EXERCISE 13.2. *Compute $H_{i,j}$ for the trefoil, see Figure 7, and more generally for the $T_{2,2n+1}$ torus knots.*

13.1. The Euler characteristic of knot Floer homology. As an obvious consequence of Theorem 13.1 we have the following

THEOREM 13.3.

$$\sum_i \sum_j (-1)^j \cdot \mathrm{rk}(H_{i,j}(K)) \cdot T^i = \Delta_K(T). \tag{7}$$

It is interesting to compare this with [**1**], [**19**], and [**6**].

13.2. Computing knot Floer homology for alternating knots. It is clear from the above formulas that if K has an alternating projection, then $F(\mathbf{x}) - G(\mathbf{x})$ is independent of the choice of state $\mathbf{x}$. It follows that if we use the chain complex associated to this Heegaard diagram, then there are no differentials in the knot Floer homology, and indeed, its rank is determined by its Euler characteristic. Indeed, by calculating the constant, we get the following result, proved in [**28**]:

THEOREM 13.4. *Let $K \subset S^3$ be an alternating knot in the three-sphere, write its symmetrized Alexander polynomial as*

$$\Delta_K(T) = \sum_{i=-n}^{n} a_i T^i$$

and let $\sigma(K)$ denote its signature. Then, $H_{i,j}(K) = 0$ for $j \neq i + \frac{\sigma(K)}{2}$, and

$$H_{i,i+\sigma(K)/2} \cong \mathbb{Z}^{|a_i|}.$$

We see that knot Floer homology is relatively simple for alternating knots. For general knots however the computation is more subtle because it involves counting holomorphic disks. In the next section we study more examples.

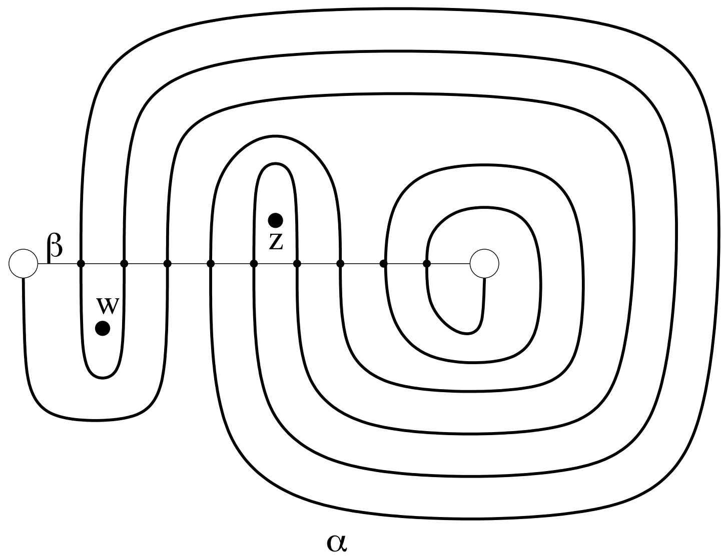

FIGURE 12.

14. More computations

For knots that admit two-pointed genus 1 Heegaard diagrams computing knot Floer homology is relatively straightforward. In this case we study holomorphic disks in the torus. For an interesting example see Figure 12. The two empty circles are glued along a cylinder, so that no new intersection points are introduced between the curve α (the darker curve) and β (the lighter, horizontal curve).

EXERCISE 14.1. *Compute the Alexander polynomial of K in Figure 12.*

EXERCISE 14.2. *Compute the knot Floer homology of K in Figure 12.*

Another special class is given by Berge knots [**2**]. These are knots that admit lens space surgeries.

THEOREM 14.3. *([**26**]) Suppose that $K \subset S^3$ is a knot for which there is a positive integer p so that p surgery $S^3_p(K)$ along K is a lens space. Then, there is an increasing sequence of non-negative integers*

$$n_{-m} < \ldots < n_m$$

with the property that $n_s = -n_{-s}$*, with the following significance. For* $-m \leq s \leq m$ *we let*

$$\delta_i = \begin{cases} 0 & \textit{if } s = m \\ \delta_{s+1} - 2(n_{s+1} - n_s) + 1 & \textit{if } m - s \textit{ is odd} \\ \delta_{s+1} - 1 & \textit{if } m - s > 0 \textit{ is even,} \end{cases}$$

Then for each s *with* $|s| \leq m$ *we have*

$$H_{n_s,\delta_s}(K) = \mathbb{Z}$$

Furthermore for all other values of i, j *we have* $H_{i,j}(K) = 0$*.*

For example the right-handed (p, q) torus knot admit lens space surgeries with slopes $pq \pm 1$, so the above theorem gives a quick computation for $H_{i,j}(T_{p,q})$.

14.1. Relationship with the genus of K. A knot $K \subset S^3$ can be realized as the boundary of an embedded, orientable surface in S^3. Such a surface is called a Seifert surface for K, and the minimal genus of any Seifert surface for K is called its *Seifert genus*, denoted $g(K)$. Clearly $g(K) = 0$ if and only if K is the unknot. The following theorem is proved in [**30**].

Theorem 14.4. *For any knot* $K \subset S^3$*, let*

$$\deg H_{i,j}(K) = \max\{i \in \mathbb{Z} \big| \oplus_j H_{i,j}(K) \neq 0\}$$

denote the degree of the knot Floer homology. Then

$$g(K) = \deg H_{i,j}(K).$$

In particular knot Floer homology distinguishes every non-trivial knot from the unknot.

For more results on computing knot Floer homology see [**33**], [**34**] [**31**] [**12**], [**32**], [**13**], and [**5**].

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